

GROUP RINGS WHICH ARE AZUMAYA ALGEBRAS

BY

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ABSTRACT. The group ring RG of a group G over a ring R (with identity $1(R)$) is a separable algebra over its center if and only if the following conditions hold:

- (a) R is a separable algebra over its center;
- (b) the center of G has finite index in G ;
- (c) the commutator subgroup G' of G has finite order m and $m1(R)$ is invertible in R .

1. Introduction. If R is a commutative ring and G a finite group of order n , then the group ring RG is a separable R -algebra if and only if $n1(R)$ is invertible in R ($1(R)$ is the identity of R). This is a well-known generalization of Maschke's theorem, a proof of which may be based on the ideas in [3, p. 41]. A ring is an *Azumaya algebra* if it is separable over its center. In particular, if $n1(R)$ is invertible in R , then RG is an Azumaya algebra. This condition is not necessary however. For example, if G is any abelian group and R any commutative ring then RG is an Azumaya algebra. A less trivial example was provided by C. T. C. Wall in [7]. He showed that RG is a direct sum of matrix rings over commutative rings when R is a ring of p -adic integers and G is a semidirect product of a finite cyclic group of order not divisible by p and a finite, abelian p -group. In particular, RG is an Azumaya algebra in this case.

The purpose of this paper is to prove the following theorem which determines which group rings RG are Azumaya algebras.

THEOREM I. *Let R be a ring with identity $1(R)$ and let G be a group. The group ring RG is an Azumaya algebra if and only if the following conditions are satisfied:*

- (a) *R is an Azumaya algebra;*
- (b) *the center of G has finite index in G ;*
- (c) *the commutator subgroup G' of G has finite order m and $m1(R)$ is invertible in R .*

It seems worthwhile to explicitly state the theorem in the classical case for comparison with Maschke's theorem.

COROLLARY. *Let F be a field and G a finite group. Then the group algebra FG is an Azumaya algebra if and only if the order of the commutator subgroup of G is invertible in F .*

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For a prime p , the condition that p does not divide the order of the commutator subgroup of the finite group G is equivalent to the condition that G is a finite group with an abelian p -Sylow subgroup and a normal p -complement. Our main result yields the following generalization of Wall's result mentioned above.

COROLLARY. *Let R be a complete local ring with residue field of characteristic $p > 0$, and let G be a finite group. Then RG is a direct sum of matrix rings over commutative rings if and only if G has an abelian p -Sylow subgroup and a normal p -complement.*

It should be remarked here that for a field F of characteristic p and a finite group G , the group algebra FG is an Azumaya algebra if and only if it is isomorphic to a direct sum of matrix rings over commutative rings.

Our main reference for the theory of separable algebras is [3]. The reader may consult [1] for additional facts about separability.

2. Preliminary reductions. In this section we reduce the proof of Theorem I to the case of group algebras over fields.

LEMMA 2.1. *Let A be a separable algebra over the commutative ring R and let ϕ be a ring epimorphism of A onto a ring B . Then ϕ maps the center of A onto the center of B ; B is a separable $\phi(R)$ -algebra and an Azumaya algebra.*

PROOF. Since ϕ is an epimorphism, $\phi(R)$ is contained in the center of B . Thus B is an R -algebra by way of ϕ . The lemma follows from Propositions II.1.11 and II.1.12 of [3].

LEMMA 2.2. *Let R be a ring with center C and G any group. Then RG is an Azumaya algebra if and only if both R and CG are Azumaya algebras.*

PROOF. Suppose RG is an Azumaya algebra. The homomorphism mapping the group G onto the group of order one induces an epimorphism of RG onto R . Hence R is an Azumaya algebra.

Let Z denote the center of CG . Then we have isomorphisms

$$RG \cong R \otimes_C CG \cong (R \otimes_C Z) \otimes_Z CG.$$

Proposition II.1.8 of [3] implies that CG is an Azumaya algebra.

Conversely, suppose both R and CG are Azumaya algebra. Then the isomorphism $RG \cong R \otimes_C CG$ implies RG is an Azumaya algebra.

This first reduction allows us to restrict our attention to the case of commutative coefficient rings.

For a ring S , let $z(S)$ denote the center of S .

LEMMA 2.3. *Let A be a commutative ring and B a subring. For any group G there is an isomorphism $AG \cong z(AG) \otimes_{z(BG)} BG$.*

PROOF. The center of AG is the free A -module generated by the class sums $c(x) = \sum g x g^{-1}$, where x is an element of G with only a finite number of G -conjugates and the sum is over the distinct conjugates. Clearly these sums also freely generate the center $z(BG)$ as a B -module. Thus $z(AG)$ can be identified with $A \otimes_B z(BG)$.

The lemma is a consequence of the following isomorphisms:

$$AG \cong A \otimes_B BG \cong (A \otimes_B z(BG)) \otimes_{z(BG)} BG \cong z(AG) \otimes_{z(BG)} BG.$$

LEMMA 2.4. *Let F and L be fields having the same characteristic and let G be any group. Then FG is an Azumaya algebra if and only if LG is an Azumaya algebra.*

PROOF. Assume FG is an Azumaya algebra and let K be the prime subfield of F . Then $z(KG)$ is a direct summand of $z(FG)$ as $z(KG)$ -modules, so it follows from Lemma 2.3 and Proposition II.1.8 of [3] that KG is an Azumaya algebra. Since F and L have the same characteristic, K may be identified with a subfield of L . Then by applying Lemma 2.3 to $K \subseteq L$ we obtain that LG is an Azumaya algebra.

DEFINITION. For a commutative ring R , the *characteristic set* $\text{CH}(R)$ is the set of nonnegative integers q such that there exists an ideal M of R with R/M an integral domain of characteristic q .

PROPOSITION 2.5. *Let R be a commutative ring and G any group. The group ring RG is an Azumaya algebra if and only if for each integer p in $\text{CH}(R)$ there is some field F of characteristic p for which FG is an Azumaya algebra.*

PROOF. Suppose RG is an Azumaya algebra. Let $D = R/M$ be a domain of characteristic p , and let F be its quotient field. Then DG is an Azumaya algebra because it is a homomorphic image of RG . After applying Lemma 2.3 to $F \supseteq D$, we obtain FG is also an Azumaya algebra. Thus the required fields exist.

Now suppose the required fields exist. In order to show RG is an Azumaya algebra, it is sufficient to show RG/MRG is separable over $z(RG)/M$ for every maximal ideal M of $z(RG)$ [3, Theorem II.7.1]. So assume M is such an ideal and let L denote the field $z(RG)/M$. The inclusion of R into $z(RG)$ implies that R maps into L , so the characteristic of L is an element of $\text{CH}(R)$. By assumption there is some field F having the same characteristic as L such that FG is an Azumaya algebra. By Lemma 2.4 LG is an Azumaya algebra.

Now let ϕ denote the composite of the following ring epimorphisms:

$$LG \rightarrow L \otimes_R RG \rightarrow L \otimes_{z(RG)} RG \rightarrow RG/MRG.$$

The last map in this sequence carries $a \otimes b$ to $a(b + MRG)$. For any x in G having finitely many G -conjugates, the class sum $c(x)$ is mapped by ϕ to $c(x) + MRG$. It follows then that $\phi(z(LG)) = (z(RG) + MRG)/MRG$, which in turn is the image of $z(RG)/M$. Since LG is separable over $z(LG)$, RG/MRG is separable over $z(RG)/M$ by Lemma 2.1. This completes the proof.

This reduces the proof of the main theorem to the case in which the coefficient ring is a field. Notice that the statement “the order of G' is invertible in R ” is equivalent to the statement “no prime divisor of the order of G' is in $\text{CH}(R)$ ”.

3. The finite case. Throughout this section we consider the group ring FG of a finite group G over a field F .

If F has characteristic zero then FG is separable over F and so FG is an Azumaya algebra. So we restrict our attention to the case in which F has characteristic $p > 0$. In view of the interpretation of the condition that p does not divide the order of G' , the result to be proved is the following.

THEOREM 3.1. *Let F be a field of characteristic $p > 0$ and G a finite group. Then FG is an Azumaya algebra if and only if G has an abelian p -Sylow subgroup and a normal p' -complement.*

PROOF. Suppose FG is an Azumaya algebra. In view of Lemma 2.4, we may assume F is a finite splitting field for all subgroups of G . Since $z(FG)$ is a commutative, finite-dimensional F -algebra, it decomposes as $z(FG) = S_1 \oplus \cdots \oplus S_n$ with each summand a finite local F -algebra. An Azumaya algebra over a finite ring is Morita equivalent to its center because the Brauer group of a finite ring is trivial [1]. Morita equivalence respects direct sums, thus each p -block FGS of FG is Morita equivalent to its center S . An Artinian local F -algebra has only one simple module and any Morita equivalent ring must have only one simple module also. Thus each p -block of FG has only one simple module. By Corollary 65.3 of [4], it follows that G has a normal p -complement N . Thus $G = NP$ with P a p -Sylow subgroup of G . It is necessary to show that P is abelian.

Since P is a homomorphic image of G , and FG is an Azumaya algebra, it follows that FP is an Azumaya algebra. The radical, $\text{rad}(FG)$, of FG is the ideal having the elements $1 - x$ as basis for x in P . By [3, Corollary II.3.7] there is an ideal I of $z(FP)$ such that $\text{rad}(FP) = IFP$. We now have the vector space sums

$$FP = F + \text{rad}(FP) = z(FP) + \text{rad}(FP) = z(FP) + IFP.$$

Since I is a nilpotent ideal, Nakayama's lemma implies $FP = z(FP)$ and so P is abelian.

The converse is well known and, as pointed out by the referee, is probably due to Morita [5]. A slightly more precise version appears in [6, p. 232]. It is shown that if $G = NP$, with N a normal p' -subgroup and P an abelian p -group, and if F is a perfect splitting field of characteristic p , then FG is isomorphic to a direct sum of matrix rings over commutative rings of the form FQ for Q a subgroup of P .

4. The infinite case. In this section we allow the group G to be infinite. The results to this point show that Theorem I will follow as a consequence of the next result.

THEOREM 4.1. *Let F be either a finite field or the field of rational numbers and let G be a group. Then FG is an Azumaya algebra if and only if:*

- (i) *the center of G has finite index in G ; and*
- (ii) *the commutator subgroup G' has finite order m , and m is invertible in F .*

The proof is given in several steps. Denote the center of G by $Z(G)$.

LEMMA 4.2. *Let $G = Z(G)H$ for some subgroup H of G and let R be any commutative ring. Then RG is an Azumaya algebra if and only if RH is an Azumaya algebra.*

PROOF. Let T be a set of elements in $Z(G)$ which represent the cosets of H in G . Any element g in G has a unique representation $g = ht$ with t in T and h in H . Moreover g and h have the same number of conjugates. If this number is finite the class sums satisfy $c(g) = c(h)t$. Since these finite class sums give an R -basis for the center, we obtain $z(RG) = \sum z(RH)t$, the sum taken over T . The coset decomposition implies $RG = \sum RHt$. From these equations we conclude $RG = z(RG) \otimes RH$,

with the tensor product over $z(RH)$. Since $z(RH)$ is a direct summand of $z(RG)$, the equivalence stated in the lemma follows from Corollaries II.1.7 and II.1.10 of [3].

LEMMA 4.3. *Let G be a group whose center has finite index. Then there is a finitely generated subgroup H of G with $G = Z(G)H$. Any such subgroup H satisfies $H' = G'$.*

PROOF. As a generating set for H , take one element from each coset of $Z(G)$ in G . If z and w are central and if x and y are arbitrary elements of G , then the commutator of zx and wy equals the commutator of x and y .

Now we begin the proof that (i) and (ii) in Theorem 4.1 imply FG is an Azumaya algebra. The proofs are different in the characteristic zero case and the characteristic nonzero case.

PROPOSITION 4.4. *If F is a field of characteristic zero and if $G/Z(G)$ is finite, then FG is an Azumaya algebra.*

PROOF. Separability over $z(FG)$ is equivalent to the existence of a separability idempotent e in $FG \otimes_{z(FG)} FG^0$ [3, Proposition II.1.1]. We construct the element. Let T be a set of representatives of the cosets of $Z(G)$ in G . Let

$$e = [G: Z(G)]^{-1} \sum t \otimes t^{-1}.$$

Then e maps to 1 under the multiplication map. It is only necessary to show $(a \otimes 1)e = (1 \otimes a)e$ for each a in FG . It is sufficient to show this for each a in G . For any t in T there is some c in $Z(G)$ and s in T with $at = sc$. Thus

$$(a \otimes 1)(t \otimes t^{-1}) = sc \otimes t^{-1} = s \otimes ct^{-1} = s \otimes s^{-1}a = (1 \otimes a)(s \otimes s^{-1}).$$

As t runs through the elements of T , so does s . It follows that e has the required properties and FG is an Azumaya algebra.

We may now restrict our attention to the case where F is the field of p elements (or any finite field of characteristic p). We assume that (i) and (ii) of Theorem 4.1 hold. In view of Lemmas 4.2 and 4.3 we may also assume that G is finitely generated.

Let M be a maximal ideal of $z(FG)$. We intend to show FG/MFG is separable over $z(FG)/M$. This is sufficient to prove FG is an Azumaya algebra. The first step is to locate a subgroup of $Z(G)$ having some special properties.

LEMMA 4.5. *In the context just described, there is a subgroup U of $Z(G)$ with the following properties:*

1. G/U is a finite group with commutator subgroup isomorphic to G' .
2. The group ring $F(G/U)$ has FG/MFG as an epimorphic image by a map which carries $z(F(G/U))$ onto $(z(FG) + MFG)/MFG$.

PROOF. Since G is finitely generated and $Z(G)$ has finite index, $Z(G)$ is a finitely generated abelian group [6, p. 177]. The structure theory for such group yields a decomposition

$$Z(G) = B \times E = B \times \langle x_1 \rangle \times \cdots \times \langle x_n \rangle$$

with B a finite group and E a free abelian group. Let T be a set of representatives for the cosets of E in G . Then T is a finite set. Let $c(t)$ denote the sum of the

G -conjugates of t . Note that all elements of G have only a finite number of conjugates because $Z(G)$ has finite index in G . Now we have a decomposition $z(FG) = \sum F E c(t)$.

The group ring FE is the F -algebra generated by the $2n$ elements x_i, x_i^{-1} , for $1 \leq i \leq n$. It follows that $z(FG)$ is a homomorphic image of a polynomial ring over F in some finite number of variables; that is, $z(FG)$ is a finitely generated F -algebra. Now let L denote the field $z(FG)/M$; L is a finitely generated algebra over F . This can happen only when L is algebraic over F [8, p. 165]. Since F is a finite field, L is also finite. The inclusion of E into $z(FG)$ induces a homomorphism of E into the multiplicative group of L . Let U denote the kernel of this homomorphism on E . Then U has finite index in E and so G/U is finite. The group U is torsion free since it is a subgroup of the torsion free group E . This implies $U \cap G' = 1$ because G' is finite. This implies $(G/U)' = UG'/U = G'$, and this proves statement 1.

The natural projection ϕ' of FG onto FG/MFG has $u - 1$ in its kernel for each u in U . Hence ϕ' factors through $F(G/U)$ and induces an epimorphism ϕ from $F(G/U)$ onto FG/MFG . In order to verify the assertion about the centers, we begin with the following observation:

If x and y are distinct conjugates in G then xU and yU are distinct conjugates in G/U .

This is an immediate consequence of the equation $U \cap G' = 1$. This implies that a class sum $c(x)$ in FG is mapped onto a class sum $c(xU)$ in $F(G/U)$ by the natural projection. It follows that $\phi(z(F(G/U))) = \phi'(z(FG))$, and statement 2 is proved.

Now the pieces may be assembled. The order of $(G/U)'$ is invertible in F by Lemma 4.5 and so $F(G/U)$ is Azumaya algebra by Theorem 3.1. Thus its epimorphic image FG/MFG is separable over the image of the center of $F(G/U)$. This image is $(z(FG) + MFG)/MFG$, which is also the image of $z(FG)/M$. Hence FG/MFG is separable over $z(FG)/M$. This completes the proof that (i) and (ii) of Theorem 4.1 imply that FG is an Azumaya algebra.

Now we begin the proof of the converse. It is assumed that F is a field and that FG is an Azumaya algebra. The proof that (i) and (ii) hold relies upon the results and ideas of Passman [6]. The first important concept is that of the finite conjugacy subgroup of G , which is denoted by $\Delta(G)$, or just Δ if there is no danger of confusion. This is the set of all elements of G which have only a finite number of conjugates. One may verify that $\Delta(G)$ is a normal subgroup of G [6, Chapter 4].

LEMMA 4.6. *If F is a field and G a group such that FG is an Azumaya algebra, then $G = \Delta(G)$, G' is finite, and the center $Z(G)$ has finite index in G .*

PROOF. Let Z denote $z(FG)$. Then FG is a finitely generated projective Z -module [3, Theorem II.3.4]. Also Z is a subring of $F\Delta$ since the class sums $c(g)$ span Z as g runs through Δ . Let T be a set of coset representations of Δ in G . Then $FG = \sum F\Delta t$ is a direct decomposition. Since FG is finitely generated over a subring of $F\Delta$, it follows that the group $D = G/\Delta$ is finite. Now there is a canonical map of FG onto FD which maps the center Z onto the center of FD by Lemma 2.1. However this map carries Δ onto 1 and thus carries Z onto F . The center of the group algebra of a finite group D has dimension one only when $D = 1$. Thus $G = \Delta$.

Now let S be a finite set of elements in G for which $FG = \sum \mathbb{Z}s$, s in S . Then $C_G(s)$ has finite index in G and, thus, $H = \bigcap C_G(s)$ also has finite index in G . It is clear that any element of H centralizes all of FG , and so $H = Z(G)$, and $Z(G)$ has finite index. Finally a lemma of Schur [6, p. 155] implies G' is finite.

All that remains to show is that the order of G' is invertible in F . There is a finitely generated subgroup H with $G = Z(G)H$, $H' = G'$ (Lemma 4.3) and FH an Azumaya algebra (Lemma 4.2). Moreover there is a torsion free subgroup E of $Z(H)$ such that H/E is finite with commutator subgroup isomorphic to G' (by the proof of Lemma 4.5). It follows that $F(H/E)$ is an Azumaya algebra and so, by Theorem 3.1, the order of G' is invertible in F .

This completes the proof of the main result.

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